AP Calculus
Review: Sections 3.1 - 3.3, Indefinite and Definite Integrals

Make sure you know the antidifferentiation rules from section 3.2 in Nutshell. You will be asked to evaluate simple indefinite integrals on the test and you cannot evaluate a definite integral if you cannot find an antiderivtaive of the integrand. The rules are listed below.

$\int k \, dx$ = kx + C This rule states that the antiderivative of a constant is the constant time x + some constant C.

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, n ≠ -1. This is the power rule for antidifferentiation.

$\int k \cdot f(x) \, dx = k \int f(x) \, dx$ This is the constant times a function rule. Note that it says that the antiderivative of a constant times a function is the constant times the antiderivative of the function.

$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$ This is the sum or difference rule. It states that the antiderivative of a sum or difference of two (or more) functions is the sum or difference of the antiderivatives of the functions.

$\int e^x \, dx = e^x + C$ This is the rule for the antiderivative of e^x.

$\int \frac{1}{x} \, dx = \ln |x| + C$ This is the rule for the antiderivative of $x^{-1} = \frac{1}{x}$, where the power rule breaks down.

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \arctan \frac{x}{a} + C, \ a \neq 0.$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \frac{1}{a} + C$$

In addition to using the elementary antiderivative rules above we also learned how to "undo" the chain rule which requires a process known as substitution. Remember that we first identify the composite function and then check to see that we have the derivative of the inside function as part of the integrand. Sometimes we have to adjust the integrand by a constant factor. When we do so we also have to multiply the integral by the reciprocal of the factor we used. In this case we change the "appearance" of the integral without changing its value (we are, after all, just multiplying by 1!) Look at the example below.

Example 1: Find the general antiderivative (indefinite integral) of $\int x^2(x^3+2)^5\,dx$

Solution: The composite function is obviously $(x^3+2)^5$. The derivative of the inside function is $3x^2$ dx. Note that we have x^2 dx so we are only off by a factor of 3. We can now multiply the integrand by 3 (inside) and multiply the integral by 1/3 (outside) and get the following.

$$\frac{1}{3}\int 3x^2(x^3+2)^5\,dx = \frac{1}{3}\,\frac{(x^3+2)^6}{6} + C = \frac{1}{18}(x^3+2)^6 + C$$

It may be easier for you at first to actually make the physical substitution. The process goes as follows.

Since the inside function is x^3+2we set this equal to some "dummy" variable, say u.

$u = x^3+2.$

We now differentiate both sides of the equation to get

$du = 3x^2\,dx$

We can now see that we need a constant factor of 3 in the integrand so we insert a 3 and multiply the outside by 1/3.

$$\frac{1}{3}\int 3x^2(x^3+2)^5\,dx$$

Now we make the substitutions.

$\frac{1}{3}\int u^5\,du$ (Since $3x^2$ dx = duand $(x^3+2)^5 = u^5$.

Now we have an integral that is in an elementary rule form. We use the simple integration rules we learned on the new integral.

$$\frac{1}{3}\int u^5\,du = \frac{1}{3}\,\frac{u^6}{6} + C = \frac{1}{18}\,u^6 + C$$

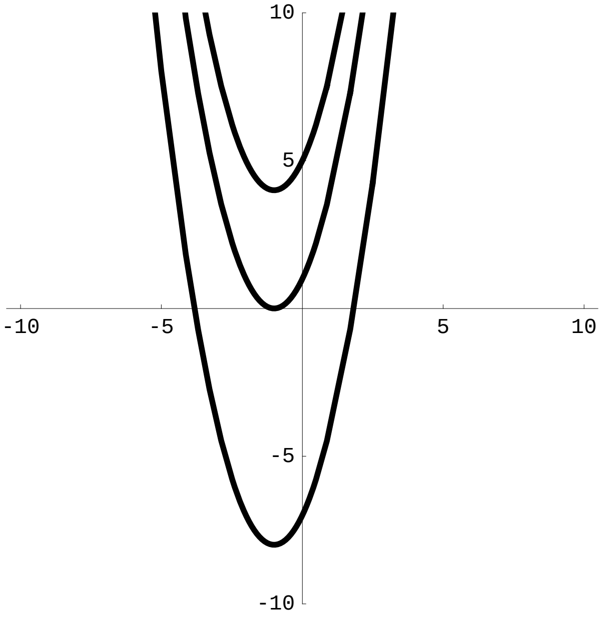
Now we just substitute back in for u.

$$\frac{1}{18}\,(x^3+2)^6 + C$$

After some practice you will probably be able to evaluate these type of integrals without the **u** substitutions but it is nice to know the process.

We know that when we evaluate an indefinite integral we add the plus C because the solution to an indefinite integral is a **family** of functions. This is because the constant of a function **disappears** when we find its derivative. The following functions all have the same derivative (and thus the same second derivative) and, therefore, will all have the same shape. They will increase and decrease on the same intervals and be concave up and concave down on the same intervals. Look at the graphs of the functions on the same set of coordinate axes.

f(x) = $x^2 + 2x + 1$, g(x) = $x^2 + 2x + 5$, h(x) = $x^2 + 2x - 7$



So how do we know which function we are dealing with? Since the functions all have the general shape they will never intersect, so if we know at least one point the graph of the function passes through we can determine which function we are looking for. This is a process known as "finding a particular solution" to a differential equation. The process requires that we know some more information about the function.

Example 2: Suppose we are given the following differential equation.

$$\frac{dy}{dx} = 2x + 2$$

and are told that y(2) = 10 (this is the same as saying that when x = 2, y = 10 or that the graph of the solution passes through the point (2, 10). We want to find the particular solution to this differential equation with the given initial conditions.

Solution: First find the general solution to the equation.

$\int (2x+2)\,dx = x^2 + 2x + C$ so we have $y = x^2 + 2x + C$

Since we also have that y(2) = 10 we can substitute in 2 for x and 10 for y.

$10 = 2^2 + 2(2) + C$ and solve for C.

$10 = 4 + 4 + C \implies C = 2.$

So the particular solution we are looking for is $y = x^2 + 2x + 2$. This is the **only** function that has the derivative 2x + 2 **and** whose graph passes through the point (2, 10).

You will also be asked to approximate left and right hand sums from a table of values and use the error formual to find the difference between the left and right hand approximations (section 3.1). You will also have to know the trapezoidal rule and the midpoint rule. Look at the example below.

Example 3: Suppose a car is moving with increasing velocity. Suppose we measure the car's velocity every two seconds, and obtain the data in the table below.

Time(sec)	0	2	4	6	8	10
Vel(ft/sec)	20	30	38	44	48	50

Approximate the total distance the car travels during the 10 second time interval. Notice that we have a rate of change (velocity) and we are trying to find the total change (the distance the car travelled). Since the velocity is always increasing the total distance travelled is the same as the displacement.

Solution: We first need to find the width of each rectangular element. In this case Δt is 2. (note that since Δt = $\frac{b-a}{n}$ in this case we have $\frac{10-0}{5} = 2$). From a table, though, it is easy to see the change in t.

So the approximations look like this:

Left[5] = Δt [f(0) + f(2) + f(4) + f(6) + f(8)]

= 2[20 + 30 + 38 + 44 + 48] = 360

Right[5] = Δt[f(2) = f(4) + f(6) + f(8) + f(10)]

= 2[30 + 38 + 44 + 48 + 50] = 420

So our left hand approximation is 360 feet and our right hand approximation is 420 feet. Since the function is increasing we expected the left hand sum to be an underestimate and our right hand sum to be on overestimate. Now let's calculate the error using the error formula.

The error formula tells us that

| Difference between upper and lower estimates | =

| Difference between f(b) and f(a) |Δt = | f(b) - f(a) |Δt

So in this case error = | 50 - 20 | · 2 = 30 · 2 = 60.

Notice that the difference between our two estimates is 420 - 360 = 60!

We could have also approximated the total distance traveled using trapezoids. The width of each trapezoid is the same width we used in the left and right hand approximations.

Δt = 2

Trap[10] = $\frac{1}{2}$ (2)[$20 + 2(30) + 2(38) + 2(44) + 2(48) + 50$] = 390

Remember that the $\frac{1}{2}$ in the formula comes from the formula for the area of a trapezoid, A = $\frac{1}{2}(h)(b_1 + b_2)$. Also, since the second base of the first trapezoid is the first base of the second we end up double counting all of the bases except the first one and the last one.

We can also approximate definite integrals using left and right sums. Look at the example below.

Example 4: Estimate $\int_0^6 (x^2+1)\,dx$ using left and right hand estimates with n = 3.

Solution: Again, we need to find the width of each rectangular element, Δx. $\Delta x = \frac{6-0}{3} = 2$.

So, Left[3] = 2[f(0) + f(2) + f(4)] = 2 [1 + 5 + 17] = 2 [23] = 46 and

Right[3] = 2[f(2) + f(4) + f(6)] = 2[5 + 17 + 37] = 2 [59] = 118

If we used the trapezoidal rule we get

$$\text{Trap }[3] = \frac{1}{2}(2)[f(0) + 2\,f(2) + 2\,f(4) + f(6)] = 1 + 10 + 34 + 37 = 82$$

If we used the midpoint rule we get

Mid[3] = 2[f(1) + f(3) + f(5)] = 2[2 + 10 + 26] = 2[38] = 76

In section 3.3 we learned how to use the Fundemental Theorem of Calculus to evaluate definite integrals. The result was the exact value of the definite integral, not an approiximation like we get uisng the left, right, trap, and midpoint rules. The Fundamental Theorem tells us that if a function is continuous on an interval [a, b] **and** we can find an antiderivative of the function on that interval then all we have to do is evaluate the antiderivative at b and a and then subtract the results.

Example 5: Evaluate $\int_0^6 (x^2+1)\,dx$. (note that this is the same integral we approximated in example 4)

since $\frac{x^3}{3} + x$ is an antiderivative of $x^2 + 1$ we get the following:

$$\int_0^6 (x^2+1)\,dx = \frac{x^3}{3} + x\,\Big|_0^6 = \frac{6^3}{3} + 6 - (0+0) = 78$$

We also combined integration by substitution and evaluating definite integrals. Look at the example below.

Example 6: Evaluate $\int_0^2 x^2(x^3+3)^4\,dx$.

The integrand is in the form of "composite function times derivative of the inside function." Only in this case part of the derivative of the inside function is missing (this is okay if the part missing is a constant factor). So, using substitution the solution looks like this

u = $x^3 + 3$ du = 3 x^2 dx and when x = 0, u = 3, when x = 2, u = 11

$$\int_0^2 x^2(x^3+3)^4\,dx \quad = \quad \mathbf{\tfrac{1}{3}}\int_{x=0}^{x=2}\mathbf{3}\,x^2(x^3+3)^4\,dx \quad = \quad \tfrac{1}{3}\int_{u=3}^{u=11}u^4\,du \quad =$$

$$\tfrac{1}{3}\,\tfrac{u^5}{5}\Big|_3^{11} = \tfrac{1}{15}[11^5 - 3^5] \approx 10,720.533$$

Of course, we could have integrated in terms of u and then substituted $x^3 + 3$ back in and evaluated the difference using 0 and 2. The results would have been the same.

Finally, we learned to define functions as definite integrals (the Second Fundamental Theorem). We learned that if *f* is a continuous interval on an interval containing *a*, then, for every *x* in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Or, in other words, if we define a function F(x) as follows

F(x) = $\int_a^x f(t)\,dt$, then F'(x) = f(x).

Example 7: Given h(x) = $\int_0^x \cos t\,dt$, $0 \le t \le 2\pi$,

(a) Find h(0)

(b) Find h(π/3)

(c) Write the equation of the tangent line to h(x) at x = π/3.

Solution:

(a) $h(0) = \int_0^0 \cos t\, dt = 0$

(b) $h(\pi/3) = \int_0^{\pi/3} \cos t\, dt = \sin t \Big|_0^{\pi/3} = \sin\frac{\pi}{3} - \sin 0 = \frac{\sqrt{3}}{3} - 0 = \frac{\sqrt{3}}{3}$

(c) $h'(\pi/3) = \cos \pi/3 = \frac{1}{2}$, so the equation of the tangent line to the graph of h at $x = \pi/3$ is

$$y - \frac{\sqrt{3}}{3} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$